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Existence of Best Approximations by Sums of Exponentials*

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In this paper we shall show that each $f \in L_p[0, 1]$ $(1 \leqslant p \leqslant \infty)$ has a best L_p approximation from the set of exponential sums, $V_n(S)$, provided S is closed. Here $V_n(S)$ denotes the set of all solutions of all n-th order linear homogeneous differential equations with constant coefficients for which the roots of the corresponding characteristic polynomial all lie in S. We thus extend the previously known existence theorems which apply only in the special cases where S is compact or where $S = \mathbb{R}$.

1. INTRODUCTION

Let S be a subset of the set $\mathbb{C}$ of complex numbers. We shall define $V_n(S)$, $n = 1, 2,\ldots$, to be the set of all complex-valued functions y defined on the interval [0, 1] which satisfy some n-th order linear homogeneous differential equation of the form

$$\{(D - \lambda_1)(D - \lambda_2) \cdots (D - \lambda_n)\}\, y(t) = 0, \qquad 0 \leqslant t \leqslant 1, \tag{1}$$

where $D = d/dt$ is the differential operator and where $\lambda_1, \lambda_2, \ldots, \lambda_n \in S$. We shall also define $V_0(S)$ to be the set whose only element is the zero function and we set

$$V_\infty(S) = \bigcup_{n=1}^{\infty} V_n(S).$$

If y satisfies (1) but does not satisfy any such differential equation of lower order we shall say that y is an exponential sum with order n. The n (not

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necessarily distinct) complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ will then be called the essential exponential parameters of y and we shall refer to the set

$$\Lambda[y] \equiv \bigcup_{i=1}^{n} \{\lambda_i\}$$

as the spectrum of y (with the null set being the spectrum of the zero function.) For example, every polynomial of degree n is an exponential sum with order $n + 1$ and with the spectrum $\{0\}$ so that $V_{n+1}(\{0\})$ is the set of all polynomials of degree at most n. The space $L_p[0, 1]$ with the associated norm, $\| \ \|_p$, will be defined in the usual manner with the understanding that the elements of $L_p[0, 1]$ may, in general, be complex-valued. Each y from $V_\infty(S)$ may then be regarded as an element of each of the spaces $L_p[0, 1]$, $1 \leqslant p \leqslant \infty$.

Our problem may now be stated as follows. Given $S \subseteq \mathbb{C}$, $1 \leqslant p \leqslant \infty$, and $f \in L_p[0, 1]$ we would like to find a best $\| \ \|_p$-approximation to f from $V_n(S)$, i.e., we would like to find some $y_0 \in V_n(S)$ such that

$$\|f - y_0\|_p = \inf\{\|f - y\|_p : y \in V_n(S)\}. \tag{2}$$

By making use of a Taylor series argument, Hobby and Rice [4] have shown that a solution to the problem exists when S is compact. In the case where $S = \mathbb{R}$, de Boor [1] and Werner [5] have independently shown that a solution exists, with both arguments making use of the fact that the approximating family $V_n(\mathbb{R})$ possesses Rice's property Z. In this paper we shall extend these results by showing that a best $\| \ \|_p$-approximation exists whenever S is closed; in subsequent work we shall consider means for characterizing and for constructing such a solution.

2. The Case of Compact S

In proving the desired existence theorem it is convenient to first establish a few preliminary results which apply when S is bounded. We shall define the seminorm $\| \ \|_{p,\alpha}$, $0 \leqslant \alpha \leqslant 1/3$, on $L_p[0, 1]$, $1 \leqslant p \leqslant \infty$, such that

$$\|f\|_{p,\alpha} = \|\chi_\alpha f\|_p \tag{3}$$

where

$$\chi_\alpha(t) = \begin{cases} 1 & \text{for} \quad \alpha \leqslant t \leqslant 1 - \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

is the characteristic function of the interval $[\alpha, 1 - \alpha]$. When S is bounded, the seminorms $\| \ \|_{p,\alpha}$, $1 \leqslant p \leqslant \infty$, $0 \leqslant \alpha \leqslant 1/3$ are actually uniformly equivalent norms on $V_n(S)$, and the differential operator D, is bounded on $V_n(S)$ as we see from the following lemma.

LEMMA 1. *Let $S \subset \mathbb{C}$ be bounded. Then there exists a constant M (depending only on S and n) such that*

$$\| D^i y \|_q \leqslant M \cdot \| y \|_{p,\alpha} \qquad i = 0, 1,..., n \tag{4}$$

holds true whenever $y \in V_n(S)$, $1 \leqslant p, q \leqslant \infty$, and $0 \leqslant \alpha \leqslant 1/3$.

Proof. In view of the monotonicity of $\| \ \|_{p,\alpha}$ with respect to both p and α it is sufficient to establish (4) when $q = \infty, p = 1$, and $\alpha = 1/3$. By enlarging S, if necessary, we may also assume that S is compact. Now given $\boldsymbol{\lambda} \in S^n$ and $\mathbf{b} \in \mathbb{C}^n$ we shall define $\mathscr{Y}_n(\mathbf{b}, \boldsymbol{\lambda})$ to be the unique solution of the differential equation (1) which satisfies the initial conditions

$$D^{i-1} y(0) = b_i, \qquad i = 1, 2,..., n. \tag{5}$$

Then $\mathscr{Y}_n(\mathbf{b}, \boldsymbol{\lambda})$ depends analytically on $\mathbf{b}$, $\boldsymbol{\lambda}$ and vanishes identically on some nondegenerate interval if and only if $\mathbf{b} = \mathbf{0}$ (cf. [2, p. 21, 75–76]. Hence if we restrict $\mathbf{b}$ to the surface, $\partial \mathbf{B}^n$, of the unit ball in $\mathbb{C}^n$ we may define

$$F_i(\mathbf{b}, \boldsymbol{\lambda}) = \| D^i \mathscr{Y}_n(\mathbf{b}, \boldsymbol{\lambda}) \|_\infty / \| \mathscr{Y}_n(\mathbf{b}, \boldsymbol{\lambda}) \|_{1,1/3} \qquad i = 0, 1, 2,..., n$$

since the denominator cannot vanish in this case. We then choose

$$M = \max\{M_0, M_1,..., M_n\}$$

where M_i is maximaum value of the continuous function F_i as $\mathbf{b}$, $\boldsymbol{\lambda}$ range over the compact sets $\partial \mathbf{B}^n$ and S^n, respectively, $i = 0, 1,..., n$. With this choice of M we see that (4) holds for all $y \in V_n(S)$. ∎

We note that a bound analogous to (4) is presented in [3, Theor. 1] within a much more general context. We also point out that the conclusion of the lemma is no longer valid when S is not bounded. For example, if

$$y_\nu(t) = \nu^{1/2} \exp(-\nu t), \qquad \nu = 1, 2,...,$$

we find that the ratio

$$\| D^i y_\nu \|_\infty / \| y_\nu \|_1 = \nu^{i+1} / [1 - \exp(-\nu)], \qquad \nu = 1, 2,...,$$

is unbounded as ν becomes infinite even when $i = 0$.

THEOREM 1. *Let S be a compact subset of $\mathbb{C}$, and let p, $1 \leqslant p \leqslant \infty$, be chosen. Then each closed $\| \ \|_p$-bounded subset of $V_n(S)$ is $\| \ \|_p$-compact, i.e., given any $\| \ \|_p$-bounded sequence $\{y_\nu\}$ from $V_n(S)$ there exists a subsequence which $\| \ \|_p$-converges to some $y \in V_n(S)$.*

Proof. In view of Lemma 1 it is sufficient to establish the theorem for the special case $p = \infty$. Let $\{y_\nu\}$ be a $\| \|_\infty$-bounded sequence from $V_n(S)$, and let $\{\mathbf{b}_\nu\}$, $\{\boldsymbol{\lambda}_\nu\}$ be chosen from $\mathbb{C}^n$, S^n, respectively, so that

$$y_\nu = \mathscr{Y}_n(\mathbf{b}_\nu, \boldsymbol{\lambda}_\nu), \qquad \nu = 1, 2, \ldots,$$

where again by $\mathscr{Y}_n(\mathbf{b}, \boldsymbol{\lambda})$ we denote the solution of the initial value problem specified by (1) and (5). Since S^n is compact, we may (by taking a subsequence, if necessary) assume that $\{\boldsymbol{\lambda}_\nu\}$ converges to some $\boldsymbol{\lambda}_0 \in S^n$. In view of Lemma 1, the $\| \|_\infty$-boundedness of $\{y_\nu\}$ implies the $\| \|_\infty$-boundedness of $\{D^i y_\nu\}$, $i = 1, 2, \ldots, n$, and therefore $\{\mathbf{b}_\nu\}$ is bounded. We may therefore (by again passing to a subsequence, if necessary) assume that $\{\mathbf{b}_\nu\}$ converges to some $\mathbf{b}_0 \in \mathbb{C}^n$. Finally, since $\mathscr{Y}_n$ depends continuously on its parameters we see that

$$y = \mathscr{Y}_n(\mathbf{b}_0, \boldsymbol{\lambda}_0)$$

is the $\| \|_\infty$-limit of $\{y_\nu\}$. ∎

Corollary 1. *Let S be a compact subset of $\mathbb{C}$, and let p, $1 \leqslant p \leqslant \infty$, be chosen. Then each $f \in L_p[0, 1]$ has a best $\| \|_p$-approximation from $V_n(S)$.*

Proof. Let $f \in L_p[0, 1]$ be chosen, and let $\{y_\nu\}$ be a $\| \|_p$-minimizing sequence for f from $V_n(S)$, i.e.,

$$\lim \|f - y_\nu\|_p = \inf\{\|f - y\|_p : y \in V_n(S)\}$$

Then $\{y_\nu\}$ is $\| \|_p$-bounded, and, in view of the theorem, we may (by passing to a subsequence, if necessary) assume that $\{y_\nu\}$ has a $\| \|_p$-limit $y \in V_n(S)$. Hence, we find

$$\|f - y\|_p \leqslant \lim [\|f - y_\nu\|_p + \|y - y_\nu\|_p] = \lim \|f - y_\nu\|_p$$

so that y is a best $\| \|_p$-approximation to f from $V_n(S)$. ∎

3. U, V, W-Sequences

We would now like to strengthen the above corollary and obtain an existence theorem when S is closed but not compact. Unfortunately, Theorem 1 cannot be extended to apply to $V_n(S)$ when S is not bounded. For example, if

$$\begin{aligned} u_\nu(t) &= \exp[-\nu t] + \exp[-\nu(1 - t)], \qquad && 0 \leqslant t \leqslant 1, \quad \nu = 1, 2, \ldots, \\ w_\nu(t) &= \sin \nu t, && 0 \leqslant t \leqslant 1, \quad \nu = 1, 2, \ldots, \end{aligned} \tag{6}$$

then $\{u_\nu\}, \{w_\nu\}$ are both $\| \|_\infty$-bounded sequences from $V_2(\mathbb{C})$, but no subsequence of either $\{u_\nu\}$ or $\{w_\nu\}$ is $\| \|_\infty$-convergent. Indeed, these two sequences illustrate fundamentally different ways in which noncompactness can be troublesome.

In order to make these intuitive ideas more precise, we shall introduce definitions for three distinct types of sequences which may be extracted from $V_n(\mathbb{C})$. We shall say that $\{y_\nu\}$ is a U-sequence, a V-sequence, or a W-sequence according as the corresponding sequence of spectral sets $\Lambda[y_\nu]$, $\nu = 1, 2, \ldots$, satisfies the respective conditions

$$\liminf\{| \operatorname{Re} \lambda | : \lambda \in \Lambda[y_\nu]\} = +\infty, \tag{7}$$

$$\sup \left\{ | \lambda | : \lambda \in \bigcup_{\nu=1}^{\infty} \Lambda[y_\nu] \right\} < +\infty, \tag{8}$$

or both of

$$\begin{aligned} &\liminf\{| \operatorname{Im} \lambda | : \lambda \in \Lambda[y_\nu]\} = +\infty, \\ &\sup \left\{ | \operatorname{Re} \lambda | : \lambda \in \bigcup_{\nu=1}^{\infty} \Lambda[y_\nu] \right\} < +\infty. \end{aligned} \tag{9}$$

Examples of U, W-sequences are provided by $\{u_\nu\}, \{w_\nu\}$, respectively, as given by (6). For large ν, the function u_ν is essentially nonzero only near the end points of $[0, 1]$ (as suggested by the U-shape), and this behavior is typical of all $\| \|_p$-bounded U-sequences which (as we shall see) converge uniformly to zero on compact subsets of $(0, 1)$. On the other hand, for large ν the function w_ν oscillates throughout the whole interval $[0, 1]$ (as suggested by the W-shape), and this behavior is typical of all W-sequences. Indeed, (as we shall see) the seminorms$\| \|_{p,\alpha}$, $0 \leqslant \alpha \leqslant 1/3$, are uniformly equivalent on the terms of a W-sequence. Finally, we note that any sequence $\{v_\nu\}$ extracted from $V_n(S)$ is a V-sequence provided that S is bounded. From Theorem 1 we see that from any $\| \|_p$-bounded V-sequence, we can extract a $\| \|_p$-convergent subsequence which, in view of Lemma 1, is also $\| \|_\infty$-convergent, $1 \leqslant p \leqslant \infty$.

Although a general sequence $\{y_\nu\}$ from $V_n(\mathbb{C})$ need not be either a U, V, or W-sequence, we may always extract from $\{y_\nu\}$ a subsequence (which we shall continue to call $\{y_\nu\}$) that may be decomposed in the form

$$y_\nu = u_\nu + v_\nu + w_\nu, \qquad \nu = 1, 2, \ldots, \tag{10}$$

where $\{u_\nu\}, \{v_\nu\}$, and $\{w_\nu\}$ are U, V, and W-sequences from $V_{n_1}(\mathbb{C})$, $V_{n_2}(\mathbb{C})$, and $V_{n_3}(\mathbb{C})$, respectively, with $n_1 + n_2 + n_3 \leqslant n$. Indeed, if $y_\nu = 0$ for infinitely many indices ν, than the zero sequence (which may be regarded as either a U, V, or W-sequence) may be used for $\{u_\nu\}, \{v_\nu\}$, and $\{w_\nu\}$ thus providing the desired decomposition of the zero subsequence of $\{y_\nu\}$. Otherwise, (by passing to a subsequence, if necessary) we may assume that $\Lambda[y_\nu]$ has exactly m, $1 \leqslant m \leqslant n$, distinct elements for each ν. We may then write

$$\Lambda[y_\nu] = \{\lambda_{1\nu}, \lambda_{2\nu}, ..., \lambda_{m\nu}\}, \qquad \nu = 1, 2, ...,$$

and (by again passing to a subsequence, if necessary) assume that for each $l = 1, 2, ..., m$ we have either

$$\lim |\operatorname{Re} \lambda_{l\nu}| = +\infty, \tag{11}$$

$$\limsup |\lambda_{l\nu}| < +\infty, \tag{12}$$

or both of

$$\begin{aligned} \lim |\operatorname{Im} \lambda_{l\nu}| &= +\infty, \\ \limsup |\operatorname{Re} \lambda_{l\nu}| &< +\infty. \end{aligned} \tag{13}$$

Upon comparing (7), (8), and (9) with (11), (12), and (13) respectively, we see that the desired decomposition follows at once from the well-known form for expressing the solutions of the differential equation (1).

With these concepts in mind, we may now establish the following lemma which precisely characterizes the important properties of U, V, W-sequences which are essential for the proof of the desired existence theorem.

LEMMA 2. *Let $\{u_\nu\}, \{v_\nu\}$, and $\{w_\nu\}$ denote U, V, and W-sequences from $V_n(\mathbb{C})$, $n = 0, 1, ...$.*

(i) *If $\{u_\nu + v_\nu + w_\nu\}$ is a $\| \; \|_p$-bounded sequence from $V_n(\mathbb{C})$ for some p, $1 \leqslant p \leqslant \infty$, then the components sequences $\{u_\nu\}, \{v_\nu\}$, and $\{w_\nu\}$ are all $\| \; \|_p$-bounded.*

(ii) *If $\{v_\nu + w_\nu\}$ is a sequence from $V_n(\mathbb{C})$, then there exists a constant $M > 0$ (depending only on the sequence) such that*

$$\| v_\nu + w_\nu \|_p \leqslant M \| v_\nu + w_\nu \|_{p,\alpha}, \qquad \nu = 1, 2, ..., \tag{14}$$

holds true for all p, α with $1 \leqslant p \leqslant \infty$ and $0 \leqslant \alpha \leqslant 1/3$ (again $\| \; \|_{p,\alpha}$ is the seminorm defined by (3)).

(iii) *If $\{u_\nu\}$ is a $\| \; \|_p$-bounded sequence for some p, $1 \leqslant p \leqslant \infty$, then*

$$\lim \| u_\nu \|_{\infty,\alpha} = 0 \tag{15}$$

for every α with $0 < \alpha \leqslant 1/3$.

(iv) *If $\{u_\nu + w_\nu\}$ is a sequence from $V_n(\mathbb{C})$ and $f \in L_p[0, 1]$ for some p, $1 \leqslant p \leqslant \infty$, then*

$$\liminf \| f + u_\nu + w_\nu \|_p \geqslant \| f \|_p . \tag{16}$$

Proof. We shall give a proof by induction on n. The lemma certainly holds when $n = 0$ since the zero sequence is the only sequence which may be extracted from $V_0(\mathbb{C})$.

We must now show that if the lemma holds in $V_{n-1}(\mathbb{C})$ then it also holds in $V_n(\mathbb{C})$, $n = 1, 2, \ldots$. In so doing it is important to note that in proving each of (i)–(iv) we lose no generality in passing to subsequences whenever it is convenient to do so. For example, if (i) fails for some sequence $\{u_\nu + v_\nu + w_\nu\}$ from $V_n(\mathbb{C})$, then (by passing to a subsequence, if necessary) we may assume that at least one of $\lim \| u_\nu \|_p$, $\lim \| v_\nu \|_p$, and $\lim \| w_\nu \|_p$ exists and is $+\infty$ so that (i) also fails for every subsequence. Thus it is sufficient to show that (i) holds for some subsequence of every given sequence $\{u_\nu + v_\nu + w_\nu\}$. Analogous considerations hold in each of (ii)–(iv). We shall divide the proof of the induction step into four sections corresponding to the statements (i)–(iv).

Section (i)

Let $\{u_\nu + v_\nu + w_\nu\}$ be a sequence from $V_n(\mathbb{C})$ with $\| \ \|_p$-bound B. We must show that some subsequence satisfies (i). Now if $u_\nu + v_\nu + w_\nu = u_\nu$ for infinitely many indices ν, then we can extract from $\{u_\nu + v_\nu + w_\nu\}$ a U-subsequence which clearly satisfies (i) (the corresponding terms v_ν, w_ν being zero for all but finitely many ν.) Analogous considerations hold if $u_\nu + v_\nu + w_\nu = v_\nu$ or $u_\nu + v_\nu + w_\nu = w_\nu$ for infinitely many ν. We may therefore restrict our attention to the remaining two cases where either $u_\nu = 0$, $v_\nu \neq 0$, $w_\nu \neq 0$ for infinitely many ν or where $u_\nu \neq 0$, $v_\nu + w_\nu \neq 0$ for infinitely many ν. Indeed, since neither of these situations can arise in $V_1(\mathbb{C})$ the validity of (i) in $V_1(\mathbb{C})$ is thereby established, and we shall assume in the remainder of this section that $n \geqslant 2$.

Suppose then that $\{v_\nu + w_\nu\}$ is a sequence from $V_n(\mathbb{C})$ with $\| \ \|_p$-bound B and that $v_\nu \neq 0$, $w_\nu \neq 0$ for each ν. Since the order of $v_\nu + w_\nu$ is at most n, and since v_ν, w_ν both have order at least 1, it follows that $v_\nu, w_\nu \in V_{n-1}(\mathbb{C})$ for all but a finite number of indices ν, and we may therefore assume that $\{v_\nu\}, \{w_\nu\}$ are V, W-sequences from $V_{n-1}(\mathbb{C})$. We shall define the auxiliary sequences

$$v_\nu^* = v_\nu / \| v_\nu \|_p, \qquad \nu = 1, 2, \ldots,$$

$$w_\nu^* = w_\nu / \| v_\nu \|_p, \qquad \nu = 1, 2, \ldots,$$

so that $\{v_\nu^*\}$ is a $\| \ \|_p$-normalized V-sequence. We may therefore assume (by passing to a subsequence, if necessary) that $\{v_\nu^*\}$ $\| \ \|_p$-converges to some $v^* \in V_{n-1}(\mathbb{C})$ with $\| v^* \|_p = 1$, and that the real sequence $\{\| v_\nu \|_p\}$ has some (possibly infinite) limit. Hence, by using the inductive hypothesis that (iv) holds in $V_{n-1}(\mathbb{C})$ we find

$$\begin{aligned} B &\geqslant \limsup \| v_\nu + w_\nu \|_p \\ &\geqslant \{\lim \| v_\nu \|_p\} \cdot \{\liminf \| v_\nu^* + w_\nu^* \|_p\} \\ &= \{\lim \| v_\nu \|_p\} \cdot \{\liminf \| v^* + w_\nu^* \|_p\} \\ &\geqslant \{\lim \| v_\nu \|_p\} \cdot \| v^* \|_p \\ &= \lim \| v_\nu \|_p \end{aligned}$$

so that $\{v_\nu\}$ and therefore $\{w_\nu\}$ must be $\| \;\|_p$-bounded. Thus (i) holds in this case.

Suppose next that $\{u_\nu + v_\nu + w_\nu\}$ is a sequence from $V_n(\mathbb{C})$ with $\| \;\|_p$-bound B, and that $u_\nu \neq 0$, $v_\nu + w_\nu \neq 0$ for each ν. Then we may assume that $\{u_\nu\}$ and $\{v_\nu + w_\nu\}$ are sequences from $V_{n-1}(\mathbb{C})$, and since (from the induction hypothesis) (ii) holds in $V_{n-1}(\mathbb{C})$ there is a constant $M > 0$ such that (14) holds with α, $0 < \alpha \leqslant 1/3$, being held fixed. Moreover, since (iii) holds in $V_{n-1}(\mathbb{C})$ we see from (15) that

$$\| u_\nu \|_{p,\alpha} \leqslant \| u_\nu \|_p/(2M)$$

for all sufficiently large ν. Using this together with (14) and the triangle inequality we find

$$\begin{aligned} \| u_\nu \|_p &\leqslant \| u_\nu + v_\nu + w_\nu \|_p + \| v_\nu + w_\nu \|_p \\ &\leqslant B + M \| v_\nu + w_\nu \|_{p,\alpha} \\ &\leqslant B + M \cdot \{\| u_\nu + v_\nu + w_\nu \|_{p,\alpha} + \| u_\nu \|_{p,\alpha}\} \\ &\leqslant B \cdot (1 + M) + \| u_\nu \|_p/2 \end{aligned}$$

for all sufficiently large ν. Hence $\{u_\nu\}$ is $\| \;\|_p$-bounded and therefore $\{v_\nu + w_\nu\}$ is also $\| \;\|_p$-bounded. But since (i) holds in $V_{n-1}(\mathbb{C})$, this implies that $\{v_\nu\}$, $\{w_\nu\}$ are individually $\| \;\|_p$-bounded and therefore (i) holds in this case also. Thus the induction step for (i) is complete.

Section (ii).

Let $\{v_\nu + w_\nu\}$ be a sequence from $V_n(\mathbb{C})$. In seeking to establish (ii) we may assume, with no loss of generality, that $\| v_\nu + w_\nu \|_\infty = 1$ for each ν. Next, if $\{\beta_\nu\}$ is any real number sequence and

$$\theta_\nu(t) = \exp(i\beta_\nu t), \qquad 0 \leqslant t \leqslant 1, \qquad \nu = 1, 2, \ldots,$$

where $i^2 = -1$, then $| \theta_\nu | \equiv 1$ so that the sequence $\{\theta_\nu[v_\nu + w_\nu]\}$ satisfies (ii) if and only if $\{v_\nu + w_\nu\}$ does. Hence, after properly selecting the phase parameters $\{\beta_\nu\}$ and factoring out the appropriate factors $\{\theta_\nu\}$ we may also assume that $\liminf \| v_\nu \|_\infty > 0$ and that $\{w_\nu\}$ is a W-sequence from $V_{n-1}(\mathbb{C})$. Moreover, since we have already shown that (i) holds in $V_n(\mathbb{C})$ the assumption that $\| v_\nu + w_\nu \|_\infty = 1, \nu = 1, 2,...,$ implies that both $\{v_\nu\}$ and $\{w_\nu\}$ are $\| \ \|_\infty$-bounded. Thus (by passing to a subsequence, if necessary) we may further assume that $\{v_\nu\}\| \ \|_\infty$-converges to some $v \in V_n(\mathbb{C})$ with $\| v \|_\infty > 0$.

Under these restrictions, we see that when $n = 1$ we have $w_\nu = 0$ for each ν so that (ii) holds by virtue of Lemma 1. On the other hand, when $n \geqslant 2$ we may use the induction hypothesis that (iv) holds in $V_{n-1}(\mathbb{C})$ together with the Hölder inequality to see that

$$\begin{aligned}
&\liminf_\nu \operatorname*{infinum}_{p,\alpha} \| v_\nu + w_\nu \|_{p,\alpha} / \| v_\nu + w_\nu \|_p \\
&\geqslant \liminf_\nu \operatorname*{infinum}_{\alpha} \| v_\nu + w_\nu \|_{1,\alpha} / \| v_\nu + w_\nu \|_\infty , \\
&= \liminf_\nu \| v_\nu + w_\nu \|_{1,1/3} , \\
&= \liminf_\nu \| v + w_\nu \|_{1,1/3} , \\
&\geqslant \| v \|_{1,1/3} > 0,
\end{aligned}$$

where the infinum is taken over the sets where $1 \leqslant p \leqslant \infty$ and $0 \leqslant \alpha \leqslant 1/3$. From this inequality we immediately infer the existence of a constant $M > 0$ such that (14) holds whenever $1 \leqslant p \leqslant \infty$ and $0 \leqslant \alpha \leqslant 1/3$. Thus the induction step is complete for (ii).

Section (iii)

First of all, since any $\| \ \|_p$-bounded U-sequence $\{u_\nu\}$ is also $\| \ \|_1$-bounded, it is sufficient to prove (iii) under the hypothesis that $p = 1$, and this being the case we may further assume that $\| u_\nu \|_1 = 1$ for each ν. We now select the sequences $\{\gamma_\nu\}$, $\{\beta_\nu\}$ from $\mathbb{C}$ such that $\beta_\nu \in \Lambda[u_\nu]$, $\nu = 1, 2,...$ (so that $\lim | \operatorname{Re} \beta_\nu | = +\infty$) and such that if

$$\theta_\nu(t) = \gamma_\nu \cdot \exp(\beta_\nu t), \qquad 0 \leqslant t \leqslant 1, \quad \nu = 1, 2,...,$$

then u_ν may be decomposed in either of the forms

$$\begin{aligned}
u_\nu &= \theta_\nu \cdot [u_\nu^* + v_\nu^* + w_\nu^*], \\
&= \theta_\nu \cdot v_\nu^* + u_\nu^{**},
\end{aligned}$$

with $\{v_\nu^*\}$ being a $\| \ \|_1$-normalized V-sequence, with $\{u_\nu^*\}$ and $\{u_\nu^{**}\} = \{\theta_\nu \cdot [u_\nu^* + w_\nu^*]\}$ being U-sequences, with $\{w_\nu^*\}$ being a W-sequence, and with the order of u_ν being the sum of the orders of either u_ν^*, v_ν^*, and w_ν^* or of v_ν^* and u_ν^{**} for each ν. Since $\{v_\nu^*\}$ is $\| \ \|_1$-normalized, we may assume (by passing to a subsequence, if necessary) that $\{v_\nu^*\}$ is $\| \ \|_1$-convergent to some $v^* \in V_n(\mathbb{C})$ with $\| v^* \|_1 = 1$. Finally, we may further assume that

$\operatorname{Re}\beta_\nu > 0$ for each ν (since in showing that (iii) holds we may always replace $u_\nu(t)$ by $u_\nu(1-t)$ for any ν) so that $\lim \operatorname{Re}\beta_\nu = +\infty$.

With these restrictions in mind, we now let α be chosen with $0 < \alpha \leqslant 1/9$. Using the inductive hypothesis that (iv) holds in $V_{n-1}(\mathbb{C})$ we find

$$\begin{aligned}
1 &= \lim \| u_\nu \|_1, \\
&\geqslant \limsup \int_{1-\alpha}^{1} | u_\nu | \, dt, \\
&\geqslant \limsup \left\{ | \theta_\nu(1-\alpha) | \cdot \int_{1-\alpha}^{1} | u_\nu^* + v_\nu^* + w_\nu^* | \, dt \right\}, \\
&\geqslant \{\limsup | \theta_\nu(1-\alpha) | \} \cdot \left\{ \liminf \int_{1-\alpha}^{1} | u_\nu^* + v_\nu^* + w_\nu^* | \, dt \right\}, \\
&\geqslant \{\limsup | \theta_\nu(1-\alpha) | \} \cdot \int_{1-\alpha}^{1} | v^* | \, dt,
\end{aligned}$$

and since $\| v^* \|_1 = 1$ we infer the existence of a constant $B > 0$ such that

$$| \theta_\nu(1-\alpha) | \leqslant B, \qquad \nu = 1, 2, \ldots .$$

This being the case, we have

$$\limsup \| \theta_\nu v_\nu^* \|_{\infty, 2\alpha} \leqslant \| v^* \|_\infty \cdot B \cdot \limsup | \exp(-\beta_\nu \alpha) | = 0$$

and since $\{u_\nu\}$ is $\| \; \|_1$-bounded this implies that $\{u_\nu^{**}\}$ is $\| \; \|_{1,2\alpha}$-bounded. But since $\{u_\nu^{**}\}$ is a U-sequence from $V_{n-1}(\mathbb{C})$ we may use the induction hypothesis that (iii) holds in $V_{n-1}(\mathbb{C})$ to see that

$$\limsup \| u_\nu \|_{\infty, 3\alpha} \leqslant \limsup \| \theta_\nu \cdot v_\nu^* \|_{\infty, 3\alpha} + \limsup \| u_\nu^{**} \|_{\infty, 3\alpha} = 0.$$

Thus the induction step is complete for (iii).

Section (iv)

We shall first establish (iv) for the special case where $u_\nu = 0$ for each ν. We may assume (by passing to a subsequence, if necessary) that $\{\| f + w_\nu \|_p\}$ has a finite limit so that $\{w_\nu\}$ is $\| \; \|_p$-bounded. We now select the real number sequence $\{\beta_\nu\}$ in such a manner that if

$$\theta_{1\nu}(t) = \exp(i\beta_{1\nu} t), \qquad 0 \leqslant t \leqslant 1, \quad \nu = 1, 2, \ldots,$$

then w_ν may be decomposed in either of the forms

$$\begin{aligned}
w_\nu &= \theta_{1\nu}[v_{1\nu} + w_{1\nu}], \\
&= \theta_{1\nu} v_{1\nu} + w_{1\nu}^*,
\end{aligned}$$

where $\{v_{1\nu}\}$ is a V-sequence and $\{w_{1\nu}\}$, $\{w_{1\nu}^*\}$ are both W-sequences from $V_{n-1}(\mathbb{C})$. Since $|\theta_{1\nu}| \equiv 1$ for each ν and since we have already shown that (i) holds in $V_n(\mathbb{C})$, the sequences $\{v_{1\nu}\}$, $\{w_{1\nu}\}$, and $\{w_{1\nu}^*\}$ are all $\| \|_p$-bounded. We may therefore assume (by passing to a subsequence, if necessary) that $\{v_{1\nu}\}$ is $\| \|_\infty$-convergent to some v_1 from $V_n(\mathbb{C})$. Next, by using the triangle inequality (and the fact $|\theta_{1\nu}| \equiv 1$ for each ν) we have

$$| \|f + \theta_{1\nu}v_1 + w_{1\nu}^*\|_p - \|f + w_\nu\|_p | \leqslant \|v_1 - v_{1\nu}\|_p$$

so that

$$\lim \|f + \theta_{1\nu}v_1 + w_{1\nu}^*\|_p = \lim \|f + w_\nu\|_p$$

and thus in showing that (iv) holds we lose no generality in assuming that $v_{1\nu} = v_1$ for each ν.

In a similar manner we decompose the $\| \|_p$-bounded sequence $\{w_{1\nu}^*\}$ from $V_{n-1}(\mathbb{C})$, and by proceeding in this manner we see that it is sufficient to establish (iv) in the special case where

$$w_\nu = \theta_{1\nu}v_1 + \theta_{2\nu}v_2 + \cdots + \theta_{k\nu}v_k, \qquad \nu = 1, 2,..., \tag{17}$$

where v_l is a fixed element of $V_{n-l+1}(\mathbb{C})$, where

$$\theta_{l\nu}(t) = \exp(i\beta_{l\nu}t), \qquad 0 \leqslant t \leqslant 1, \quad \nu = 1, 2,..., \tag{18}$$

and where $\{\beta_{l\nu}\}$ is a real number sequence with

$$\lim |\beta_{l\nu}| = +\infty \tag{19}$$

for each $l = 1, 2,..., k$ with $k \leqslant n$.

Finally, since the class of finite linear combinations of characteristic functions of subintervals of [0, 1] is dense in $L_p[0, 1]$, it is sufficient to establish (iv) whenever f, $v_1, v_2,..., v_k$ are all constant multiples of the characteristic function of a single subinterval of [0, 1], or more simply, whenever f, $v_1, v_2,..., v_k$ are all complex constants (with this last step involving a simple change of variables, if necessary.) But in this special situation we may use (17)–(19) together with Hölder's inequality to obtain

$$\begin{aligned}
\lim \|f + w_\nu\|_p &\geqslant \liminf \int_0^1 \left| f + \sum_{l=1}^k v_l\theta_{l\nu} \right| dt, \\
&\geqslant \liminf \left| \int_0^1 \left[f + \sum_{l=1}^k v_l\theta_{l\nu} \right] dt \right|, \\
&= \liminf \left| f + \sum_{l=1}^k v_l[\theta_{l\nu}(1) - 1]/\beta_{l\nu} \right|, \\
&= |f| = \|f\|_p .
\end{aligned}$$

Thus we have shown that (iv) holds in $V_n(\mathbb{C})$ in the special case where $u_\nu = 0$ for each ν.

We shall now remove this restriction. Indeed, in the general case we may again assume (by passing to a subsequence if necessary) that $\{\| f + u_\nu + w_\nu \|_p\}$ has a finite limit. Since we have shown that (i) holds in $V_n(\mathbb{C})$ this implies that $\{u_\nu\}$ and $\{w_\nu\}$ are both $\| \ \|_p$-bounded. Finally, by using the fact that (iii) and the above special case of (iv) all hold in $V_n(\mathbb{C})$ we have

$$\begin{aligned}\lim \| f + u_\nu + w_\nu \|_p &\geqslant \liminf \| f + u_\nu + w_\nu \|_{p,\alpha} \\ &\geqslant \liminf \{\| f + w_\nu \|_{p,\alpha} - \| u_\nu \|_{p,\alpha}\} \\ &= \liminf \| f + w_\nu \|_{p,\alpha} \\ &\geqslant \| f \|_{p,\alpha}\end{aligned}$$

for all α, $0 < \alpha \leqslant 1/3$, and from the arbitrariness of α we conclude that (16) must hold. This then finishes the induction for (iv) and so completes the proof of the lemma. ∎

It would be desirable to simplify the rather long and tedious proof of Lemma 2, and since (i), (iv) are the only sections needed for the desired existence theorem it would be nice to prove these results independently. Unfortunately, the above induction proceeds cyclically (i.e., to prove that (i) holds in $V_n(\mathbb{C})$ we make use of the fact that (i)–(iv) all hold in $V_{n-1}(\mathbb{C})$, and then use the fact that (i) holds in $V_n(\mathbb{C})$ in showing that (iii)–(iv) also hold in $V_n(\mathbb{C})$, etc.), and we have been unsuccessful in proving (i), (iv) by any other argument.

4. The Case of Closed S

Using Lemma 2 we may now prove the following basic existence theorem.

Theorem 2. *Let $S \subseteq \mathbb{C}$, $1 \leqslant p \leqslant \infty$, and $n = 1, 2, \ldots$. Then every $f \in L_p[0, 1]$ has a best $\| \ \|_p$-approximation from $V_n(S)$ if and only if S is closed.*

Proof. Let S be closed, let $f \in L_p[0, 1]$, and let $\{y_\nu\}$ be chosen from $V_n(S)$ in such a manner that

$$\lim \| f - y_\nu \|_p = \text{infinum}\{\| f - y \|_p : y \in V_n(S)\}.$$

We may assume (by passing to a subsequence, if necessary) that $\{y_\nu\}$ may be decomposed in such a manner that

$$y_\nu = u_\nu + v_\nu + w_\nu, \qquad \nu = 1, 2, \ldots,$$

where $\{u_\nu\}$, $\{v_\nu\}$, $\{w_\nu\}$ are U, V, W-sequences, respectively, from $V_n(S)$. Since $\{y_\nu\}$ is $\| \ \|_p$-bounded, we infer from Lemma 2(i) that the component sequences $\{u_\nu\}$, $\{v_\nu\}$, $\{w_\nu\}$ are all $\| \ \|_p$-bounded, so that in view of Theorem 1 we may further assume (by again passing to a subsequence, if necessary) that $\{v_\nu\}$ $\| \ \|_p$-converges to some $v \in V_n(S)$. This being the case we may use Lemma 2(iv) to obtain.

$$\lim \|f - y_\nu\|_p = \lim \|f - v - u_\nu - w_\nu\|_p \geqslant \|f - v\|_p$$

so that v is a best $\| \ \|_p$-approximation to f from $V_n(S)$. Thus the closure of S is a sufficient condition for every $f \in L_p[0, 1]$ to have a best $\| \ \|_p$-approximation from $V_n(S)$.

On the other hand, the necessity of this condition is apparent (e.g., if $\lambda \notin S$ is a limit point of S, then the function

$$f(t) = \exp(\lambda t), \qquad 0 \leqslant t \leqslant 1$$

has no best $\| \ \|_p$-approximation from $V_n(S)$) so that the proof is complete. ∎

COROLLARY 1. *Let $S \subseteq \mathbb{C}$ be closed, let $1 \leqslant p \leqslant \infty$, and let $n = 1, 2, \ldots$. Then every $f \in L_p[0, 1]$ has a best $\| \ \|_p$-approximation from the set $V_n{}^r(S)$ of all real-valued exponential sums y contained in $V_n(S)$.*

Proof. Replace $V_n(S)$ by $V_n{}^r(S)$ in the proof of the theorem. ∎

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